INTRODUCTION

In the present paper we consider conformal Killing, closed conformal Killing, coclosed conformal Killing and harmonic forms which are defined on Riemannian globally symmetric spaces. In particular, we prove vanishing theorems for conformal Killing $L^2$-forms on a Riemannian globally symmetric space of noncompact type. In addition, we prove vanishing theorems for harmonic forms on some Riemannian globally symmetric spaces of compact type. Our proofs are based on the Bochner technique and its generalized version that are most elegant and important analytical methods in differential geometry “in the large”.

The results of the present paper were announced at the International Conference "Differential Geometry" organized by the Banach Center from June 18 to June 24, 2017 at Będlewo (Poland) and at the International Conference "Modern Geometry and its Applications" dedicated to the 225th anniversary of the birth of N.I. Lobachevsky and organized by the Kazan Federal University from November 27 to December 3, 2017 at Kazan (Russian Federation).

PRELIMINARIES

More then thirty years ago Bourguignon has investigated (see [1]) the space of natural (with respect to isometric diffeomorphisms) differential operators of order one determined on vector bundle $\Lambda^r M$ of exterior differential $r$-forms and taking their values in the space of homogeneous tensor fields on $(M, g)$.

Bourguignon has proved the existence of three basis natural operators of this space, but only the following two $D_1$ and $D_2$ of them were recognized. The first operator $D_1$ is the exterior differential operator $d : C^\infty \Lambda^r M \to C^\infty \Lambda^{r+1} M$ and the second operator $D_2$ is the exterior co-differential operator $d^* : C^\infty \Lambda^r M \to C^\infty \Lambda^{-r} M$.

About the third basis natural operator $D_3$, it was said that except for case $r = 1$, this operator does not have any simple geometric interpretation. Next, for the case $r = 1$, it was explained that the kernel of this operator consists of infinitesimal conformal transformations on $(M, g)$.

In connection with this, we have received a specification of the Bourguignon proposition and proved (see [4]) that the basis of natural differential operators consists of three operators of following forms:

$$D_1 = \frac{1}{r!} d; \quad D_2 = \frac{1}{r!} g \wedge d^*;$$

$$D_3 = \nabla - \frac{1}{r!} d - \frac{1}{r!} g \wedge d^*$$

where

$$\left(g \wedge d^* \omega\right) (X_0, X_1, ..., X_r) =$$

$$= \sum_{a=2}^{r} (-1)^a g(X_{0}, X_a) \left(d^* \omega\right)(X_1, ..., X_{a-1}, X_{a+1}, ..., X_r)$$


Copyright © 2017, Alexandrova et al. This is an open access article distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), allowing third parties to copy and redistribute the material in any medium or format and to remix, transform, and build upon the material for any purpose, even commercially, provided the original work is properly cited and states its license. This article is published with open access at http://science-evolution.ru/.
for an arbitrary exterior differential r-form \( \omega \) and any vector fields \( X_1, X_2, \ldots, X_r \) on \( M \).

The kernel of \( D_1 \) consists of closed exterior differential r-forms, the kernel of \( D_2 \) consists of co-closed exterior differential r-forms and kernel of \( D_3 \) consists of conformal Killing r-forms or, in other words, conformal Killing-Yano tensors of order \( r \) that constitute three vector spaces \( D'(M, \mathbb{R}) \), \( F'(M, \mathbb{R}) \) and \( T'(M, \mathbb{R}) \) respectively. These vector spaces are subspaces of the vector space of exterior differential r-forms on \( (M, g) \) which we denote by \( \mathfrak{X}(M, \mathbb{R}) \).

**Remark.** The concept of conformal Killing tensors was introduced by S. Tachibana about forty years ago (see [5]). He was the first who has generalized some results of a conformal Killing vector field (or, in other words, an infinitesimal conformal transformation) to a skew symmetric covariant tensor of order 2 named him the conformal Killing tensor. Kashiwada has generalized this concept to conformal Killing forms of order \( r \geq 2 \) (see [18]). The theory of conformal Killing forms is contained in the monographs [9] and [15]. In addition, there are many various applications of these tensors in theoretical physics (see, for example, [3]; [4]; [9]; [10]; [20]; [21]; [22]). In particular, we have proved that the vector space \( T'(M, \mathbb{R}) \) on a compact n-dimensional \((1 \leq r \leq n-1)\) Riemannian manifold \((M, g)\) is finite-dimensional. In addition, the number \( t_r(M) = t'(M, \mathbb{R}) \) is a conformal invariant \( M \) of \((M, g)\) such that \( t_r(M) = t_r(M) \).

The condition \( \omega \in \ker D_1 \cap \ker D_2 \) characterizes the form \( \omega \) as a closed conformal Killing form or closed conformal Killing-Yano tensor (see, for example, [9, pp. 416]; [23]). Sometimes, closed conformal Killing forms are also called planar forms (see [16]). Therefore, the space of all closed conformal Killing r-forms we can define as \( P'(M, \mathbb{R}) = T'(M, \mathbb{R}) \cap D'(M, \mathbb{R}) \).

**Remark.** The concept of Killing tensors was introduced by K. Yano about fifteen years ago (see [19]). He was the first who has generalized some results of a Killing vector field (or, in other words, an infinitesimal isometric transformation). In turn, planar forms generalized the concept of concircular vector fields. In addition, we have proved that the vector spaces \( K'(M, \mathbb{R}) \) and \( P'(M, \mathbb{R}) \) on a compact n-dimensional \((1 \leq r \leq n-1)\) Riemannian manifold \((M, g)\) are finite-dimensional. Moreover, the numbers \( k_r(M) = K'(M, \mathbb{R}) \) and \( p_{r} (M) = P'(M, \mathbb{R}) \) are conformal invariant of \((M, g)\) such that \( k_r(M) = p_{r} (M) \) (see [15] and [16]).

The condition \( \omega \in \ker D_1 \cap \ker D_2 \) characterizes the form \( \omega \) as a harmonic form (see [19]; [24, pp. 107–113]). Hence, the space of all harmonic r-forms we can define as

\[
H'(M, \mathbb{R}) = D'(M, \mathbb{R}) \cap F'(M, \mathbb{R})
\]

On a closed and oriented Riemannian manifold \((M, g)\) the condition

\[
\omega \in \ker d \cap \ker d^* \text{ is equivalent to the following condition } \omega \in \ker \Delta \text{ (see, for example, [37 pp. 202]).}
\]

**Remark.** Harmonic forms are a classical object of investigation of differential geometry for the last seventy years, beginning with the well-known scientific works of Hodge (see, for example, [24, pp. 107–113] and [25]). It is well known by Hodge that the vector space \( H'(M, \mathbb{R}) \) of harmonic r-forms on a compact \( n\)-dimensional \((1 \leq r \leq n-1)\) Riemannian manifold \((M, g)\) is finite-dimensional. Moreover, the dimension of \( H'(M, \mathbb{R}) \) equals to the Betti number \( b_r(M) \) of \((M, g)\) such that \( b_r(M) = b_{r-n}(M) \) (see [2, pp. 202–208; 385–391]).

In conclusion, we denote by \( C'(M, \mathbb{R}) \) the vector space of parallel or covariantly constant r-forms on \( M \) with respect to \( \nabla \), i.e.,

\[
C'(M, \mathbb{R}) = T'(M, \mathbb{R}) \cap D'(M, \mathbb{R}) \cap F'(M, \mathbb{R})
\]

**Riemannian symmetric spaces** are also a classical object of investigation of differential geometry as differential forms. Cartan obtained the basic theory of symmetric spaces between 1914 and 1927. Riemannian symmetric spaces have been studied by many other authors. In particular, beginning in the 1950s, Harish-Chandra, Helgason, and others developed harmonic analysis and representation theory on these spaces and their Lie groups of isometries (see, for example, [2, pp. 235–264]; [13]; [17]; [26]; [38]; [39, pp. 222–292]).

We recall here that the Riemannian symmetric space is a finite dimensional Riemannian manifold \((M, g)\), such that for every its point \( x \) there is an involutive geodesic symmetric \( s_x \), such that \( x \) is an isolated fixed point of \( s_x \). \((M, g)\) is said to be Riemannian locally symmetric if its geodesic symmetries are in fact isometric. This is equivalent to the vanishing of the covariant derivative of the curvature tensor \( R \) of \((M, g)\) (see [39, p. 244]).

A Riemannian locally symmetric space is said to be a Riemannian globally symmetric space if, in addition, its geodesic symmetries are defined on all \((M, g)\). A Riemannian globally symmetric space is complete (see [39, p. 244]). In addition, a complete and simply connected Riemannian locally symmetric space is a Riemannian globally symmetric space (see [39, p. 244]).

Riemannian globally symmetric spaces can be classified by classifying their isometry groups. The classification distinguishes three basic types of Riemannian globally symmetric spaces: spaces of so-called compact type, spaces of so-called non-compact type and spaces of Euclidean type (see, for example, [26; p. 207–208]; [39, p. 252]). An addition, if \((M, g)\) is a Riemannian globally symmetric spaces of compact type then \((M, g)\) is a compact Riemannian manifold with non-negative sectional curvature and positive-definite Ricci tensor (see [39, p. 256]).

Let \((M, g)\) be a Riemannian locally symmetric space, then \( \nabla Ric = 0 \) for the Ricci tensor \( Ric \) of \((M, g)\). If, in addition, \((M, g)\) is irreducible, then it is Einstein (see

50
[2, p. 254]). If in this case, the Einstein constant is positive, then it follows from Myer’s theorem (see [2, p. 171]) that the space \((M,g)\) is compact. Then one can show that the curvature operator is nonnegative (see also [2, p. 254]). Therefore, \((M,g)\) is a Riemannian globally symmetric space of compact type. If, in addition, \((M,g)\) is simply-connected and its curvature operator is positive-definite, then it is a Euclidian sphere (see also [2, p. 228]). Therefore, we can conclude that a simply-connected and irreducible Riemannian locally symmetric space with positive-definite curvature operator is a Euclidian sphere.

On the other hand, if \((M,g)\) is a Riemannian globally symmetric spaces of non-compact type then \((M,g)\) is a complete non-compact Riemannian manifold with non-positive sectional curvature and negative-definite Ricci tensor, and diffeomorphic to a manifold with non-positive sectional curvature and symmetric space of noncompact type that is a complete non-compact Riemannian manifold with non-negative sectional curvature.

If a Riemannian symmetric space of non-compact type has an even dimension then the following theorem on conformal Killing \(L^2\)-forms is true.

**Theorem 2.** Let \((M,g)\) be a 2r-dimensional \((m \geq 2)\) simply connected Riemannian symmetric space of non-compact type. Then every conformal Killing \(L^2\)-form of degree \(r\) is parallel form on \((M,g)\). If the volume of \((M,g)\) is infinite, then every conformal Killing \(L^2\)-form of degree \(r\) is identically zero on \((M,g)\).

**Proof.** In our paper [31] we have proved that an arbitrary conformal Killing \(L^2\)-form of degree \(r\) is parallel form on 2r-dimensional complete non-compact Riemannian manifold \((M,g)\) with negative semi-definite curvature operator. It means that \(C'(M,\mathbb{R}) = T'(M,\mathbb{R})\). If the volume of \((M,g)\) is infinite, then every conformal Killing \(L^2\)-form of degree \(r\) is identically zero on \((M,g)\).

In this case, Theorem 2 is a corollary of this proposition.

An important invariant of a symmetric space is its rank which is the maximal dimension of a totally geodesic flat subspace. In particular, rank-one symmetric spaces are an important class among symmetric spaces (see [16] and [17]). In his book [17], Chavel gave a beautiful account of the rank-one symmetric spaces from a geometric point of view up to the classification of them, which he left for the reader to pursue as a matter in Lie group theory. We prove the following theorem for conformal Killing forms on real rank-one symmetric spaces using his results.

**Theorem 3.** Let \((M,g)\) be a simply connected symmetric space of rank one type. If, in addition, \((M,g)\) is a space of compact type, and of odd dimension \(n = 2k + 1\) for \(k \geq 1\), then \(t_r(M) = (n + 2)r!(r + 1)(n - r + 1)\), \(k_r(M) = (n + 1)r!(r + 1)(n - r)\), and \(p_r(M) = (n + 1)!r!(n - r + 1)\) for an arbitrary \(1 \leq r \leq 2k\).

**Proof.** We consider a rank-one Riemannian symmetric space \((M,g)\) of compact type. If, in addition, \((M,g)\) is a simply connected manifold of odd dimension \(n = 2k + 1\) then \((M,g)\) is a sphere of constant sectional curvature (see [13]). The vector spaces over \(\mathbb{R}\) of conformal Killing, coclosed and closed conformal Killing \(p\)-forms on an Euclidian \(n\)-sphere have finite dimensions which are equal to \(t_r(M) = (n + 2)r!(r + 1)(n - r + 1)\), \(k_r(M) = (n + 1)r!(r + 1)(n - r)\), and \(p_r(M) = (n + 1)!r!(n - r + 1)\) respectively (see [15] and [16]). This proves our Theorem 3.

\[\text{Vol}_g(M)\] of \((M,g)\) is infinite then we obtain a contradiction with our condition that \(\omega \in L^2(M^\prime)\).

It remains to recall that a Riemannian globally symmetric spaces of non-compact type \((M,g)\) is a complete non-compact Riemannian manifold with non-positive sectional curvature.

**Proof.** In our paper [31] we have proved that an arbitrary conformal Killing \(L^2\)-form of degree \(r\) is parallel form on 2r-dimensional complete non-compact Riemannian manifold \((M,g)\) with negative semi-definite curvature operator. It means that \(C'(M,\mathbb{R}) = T'(M,\mathbb{R})\). If the volume of \((M,g)\) is infinite, then every conformal Killing \(L^2\)-form of degree \(r\) is identically zero on \((M,g)\).

In this case, Theorem 2 is a corollary of this proposition.

An important invariant of a symmetric space is its rank which is the maximal dimension of a totally geodesic flat subspace. In particular, rank-one symmetric spaces are an important class among symmetric spaces (see [16] and [17]). In his book [17], Chavel gave a beautiful account of the rank-one symmetric spaces from a geometric point of view up to the classification of them, which he left for the reader to pursue as a matter in Lie group theory. We prove the following theorem for conformal Killing forms on real rank-one symmetric spaces using his results.

**Theorem 3.** Let \((M,g)\) be a simply connected symmetric space of rank one type. If, in addition, \((M,g)\) is a space of compact type, and of odd dimension \(n = 2k + 1\) for \(k \geq 1\), then \(t_r(M) = (n + 2)r!(r + 1)(n - r + 1)\), \(k_r(M) = (n + 1)r!(r + 1)(n - r)\), and \(p_r(M) = (n + 1)!r!(n - r + 1)\) for an arbitrary \(1 \leq r \leq 2k\).

**Proof.** We consider a rank-one Riemannian symmetric space \((M,g)\) of compact type. If, in addition, \((M,g)\) is a simply connected manifold of odd dimension \(n = 2k + 1\) then \((M,g)\) is a sphere of constant sectional curvature (see [13]). The vector spaces over \(\mathbb{R}\) of conformal Killing, coclosed and closed conformal Killing \(p\)-forms on an Euclidian \(n\)-sphere have finite dimensions which are equal to \(t_r(M) = (n + 2)r!(r + 1)(n - r + 1)\), \(k_r(M) = (n + 1)r!(r + 1)(n - r)\), and \(p_r(M) = (n + 1)!r!(n - r + 1)\) respectively (see [15] and [16]). This proves our Theorem 3.

\[\text{Vol}_g(M)\] of \((M,g)\) is infinite then we obtain a contradiction with our condition that \(\omega \in L^2(M^\prime)\).

It remains to recall that a Riemannian globally symmetric spaces of non-compact type \((M,g)\) is a complete non-compact Riemannian manifold with non-positive sectional curvature.
HARMONIC FORMS ON A RIEMANNIAN
GLOBALLY SYMMETRIC SPACES

In this section we consider harmonic forms on a Riemannian globally symmetric space of compact type. In the simply connected case, “compact type” is equivalent to the compactness condition that was considered in Section 4 in [41].

First we prove the following theorem for harmonic forms on a Riemannian symmetric space of compact type that is a compact Riemannian manifold with non-negative sectional curvature and positive-definite Ricci tensor. Side by side, its curvature operator $R$ is nonnegative (see [35]). In this case, the Bochner technique tells us that all harmonic $r$-forms ($2 \leq r \leq n - 1$) are parallel and 1-forms are identically zero (see [2, pp. 208; 212; 221]). Now, a parallel form is necessarily invariant under the holonomy. Thus, we are left with a classical invariance problem (see [37, pp. 306-307]). In this case, the Betti numbers $b_i(M) = b_{n-i}(M)$ are parallel and 1-forms vanish everywhere and every harmonic $r$-form ($r \geq 2$) is parallel. In the case of odd dimension, then its Betti numbers $b_i(M) = b_{n-i}(M) = 1$. We proved the following theorem.

**Theorem 4.** Let $(M, g)$ be an $n$-dimensional ($n \geq 2$) non-compact simply connected Riemannian globally symmetric space of compact type. Then all harmonic one-forms vanish everywhere and every harmonic $r$-form ($r \geq 2$) is parallel. In this case, the Betti numbers $b_i(M) = b_{n-i}(M) = 0$ and $b_r(M) \leq \binom{n}{r}$ for any $r = 2, ..., n - 2$ (see [2, p. 212]).

In conclusion, we can formulate an obvious statement.

**Theorem 5.** Let $(M, g)$ be an $n$-dimensional simply connected symmetric space of rank one. If, in addition, $(M, g)$ is a manifolds of compact type, and of odd dimension, then its Betti numbers $b_1(M) = b_n(M) = 1$.

On the other hand, if $(M, g)$ is a real Riemannian symmetric space of non-compact type, and of rank one then it is one of the four spaces: $S^n, \mathbb{RP}^n, \mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^n$. The final is the 16-dimensional Cayley plane. Accordingly, compact-type symmetric spaces of rank-one have strictly positive sectional curvature. In the case of odd dimension $(M, g)$ is a Euclidian sphere (see [16]). On the other hand, it is well known that a compact Riemannian manifold $(M, g)$ with positive constant sectional curvature admits no nonzero harmonic forms (see [2, p. 212]). Therefore, its Betti numbers $b_1(M) = 0$ and $b_1(M) = 1$.

In conclusion, we can formulate an obvious statement.

**Theorem 5.** Let $(M, g)$ be an $n$-dimensional simply connected symmetric space of rank one. If, in addition, $(M, g)$ is a manifolds of compact type, and of odd dimension, then its Betti numbers $b_1(M) = b_n(M) = 1$.

On the other hand, if $(M, g)$ is a simply connected symmetric space of rank one, and of non-compact type, then $(M, g)$ carries no harmonic $L^2$-forms except when $r = n/2$ in which case $H^r(M, \mathbb{R})$ is infinite dimensional.

**Proof.** Chavel has proved in [17] that if $(M, g)$ is a rank-one Riemannian symmetric space of compact type then it is one of the four spaces: $S^n, \mathbb{RP}^n, \mathbb{CP}^n, H\mathbb{P}^n$ and $O\mathbb{P}^n$. The final is the 16-dimensional Cayley plane. Accordingly, compact-type symmetric spaces of rank-one have strictly positive sectional curvature. In the case of odd dimension $(M, g)$ is a Euclidian sphere (see [16]). On the other hand, it is well known that a compact Riemannian manifold $(M, g)$ with positive constant sectional curvature admits no nonzero harmonic forms (see [2, p. 212]). Therefore, its Betti numbers $b_1(M) = 0$ and $b_1(M) = 1$.

In conclusion, we can formulate an obvious statement.

**Theorem 5.** Let $(M, g)$ be an $n$-dimensional simply connected symmetric space of rank one. If, in addition, $(M, g)$ is a manifolds of compact type, and of odd dimension, then its Betti numbers $b_1(M) = b_n(M) = 1$.

On the other hand, if $(M, g)$ is a simply connected symmetric space of rank one, and of non-compact type, then $(M, g)$ carries no harmonic $L^2$-forms except when $r = n/2$ in which case $H^r(M, \mathbb{R})$ is infinite dimensional (see [32]). This completes the proof of Theorem 5.

**ACKNOWLEDGMENTS**

Our work was supported by Russian Foundation for Basis Research (projects Nos. 16-01-00053-a and 16-01-00053-a).

**REFERENCES**


